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MINIMUM CONVEX-COST FLOWS IN NETWORKS

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. In this Memorandum the author shows that several problems which are usually treated by other methods can be regarded as problems of minimum cost flows, and that an algorithm of shortest-path type can be used to solve all of them.

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SUMMARY

An algorithm is given for solving minimum cost flow problems where the shipping cost over an arc is a convex function of the number of units shipped along that arc. This provides a unified way of looking at many seemingly unrelated problems in different areas. For example, problems associated with electrical networks, with increasing the capacity of a network under a fixed budget, with Laplace equations, and with the Max-Flow Min-Cut Theorem may all be formulated into minimum convex-cost flow problems.

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CONTENTS

MINIMUM CONVEX-COST FLOWS IN NETWORKS

1. INTRODUCTION

Consider a connected network consisting of nodes N_i and arcs A_{ij} leading from N_i to N_j. Among the nodes N_i, there is a special node N_s called the source, and a special node N_t called the sink. The flow from N_i to N_j in the arc A_{ij} is denoted by x_{ij}. We consider the following problem:

$$\text{(1)} \qquad \text{Min } z = \Sigma\, c_{ij}(x_{ij}) ,$$

subject to

$$\text{(2)} \qquad \sum_i x_{ij} - \sum_k x_{jk} = \begin{cases} -v & \text{for } j = s , \\ 0 & \text{for } j \neq s,t , \\ v & \text{for } j = t , \end{cases}$$

where $c_{ij}(x_{ij})$ are nonnegative convex functions of x_{ij}, and the arc flows x_{ij} are required to be positive integers or zero. Note that Eqs. (2) express the conservation of flow at nodes other than the source and the sink, and that the objective function (1) is a sum of convex functions (not necessarily strictly convex), and is thus convex.

We shall discuss in Sec. 3 how this problem is related to problems of finding maximum flow in a network with arc-capacity restrictions, problems in the synthesis of traffic or communication systems, electrical-network problems, and certain boundary-value problems. Similar work has been done in this area (See [1],[3],[7], and [10]). In [3], linear

cost functions c_{ij} (x_{ij}) are considered; the method of [1] deals with bipartite networks and starts with a feasible solution.

The algorithm presented in this paper deals with general convex-cost functions in an arbitrary network and gives a feasible solution which is optimal for the parameter v. In spirit, it is closely related to that of [1] and [3].

A set of positive x_{ij} satisfying (2) is called a flow pattern with value v. A flow pattern which minimizes (1) for fixed v is called an optimal flow pattern corresponding to v. Since the cost of shipping the flow along an arc is a convex function of the amount of flow shipped, the cost of shipping one additional unit of flow along the arc will depend on how much flow already exists on the arc. Following Beale [1], we define the so-called "up-costs" of an arc as follows. For an arc A_{ij} with $x_{ij} \geq 0$ in the arc, the up-cost of that arc is the cost of sending one additional unit of flow from N_i to N_j, i.e.,

(3) $$u_{ij}(x_{ij}) = c_{ij}(x_{ij} + 1) - c_{ij}(x_{ij}) \quad \text{for } x_{ij} \geq 0 .$$

If we want to send one unit of flow from N_j to N_i where there already exists $x_{ij} \geq 0$ in the arc, then the one additional unit of flow from N_j to N_i will cancel one unit of x_{ij} without affecting the value v in (1). Hence, the cost of sending one unit of flow from N_j to N_i, where there already exists $x_{ij} \geq 0$, is actually negative.

We shall call this the "down-cost" of an arc, i.e., the cost of sending one unit of flow from N_j to N_i. In symbols,

(4) $$d_{ji}(x_{ij}) = c_{ij}(x_{ij}) - c_{ij}(x_{ij} - 1) \quad \text{for } x_{ij} \geq 1.$$

We shall assume throughout the Memorandum that $c_{ij}(0) \equiv 0$; then it follows from the convexity of the cost functions that

$$c_{ij}(o) + [c_{ij}(x_{ij} + 1) - c_{ij}(0)]x_{ij}/(x_{ij} + 1) > c_{ij}(x_{ij}) \quad \text{for } x_{ij} > 0.$$

Since $c_{ij}(0) = 0$, we have

$$c_{ij}(x_{ij} + 1) \geq (x_{ij} + 1)\, c_{ij}(x_{ij})/x_{ij} \ .$$

This means the up-cost of an arc is always positive. Similar reasoning shows that the down-cost of an arc is always negative. Furthermore, for any tw, nonnegative integers a and b with $a < b$, we have

(5) $$u_{ij}(a) \leq u_{ij}(b) \ ,$$

(6) $$d_{ji}(a) \leq d_{ji}(b) \ .$$

For a given network, let a flow pattern with value v_1 be given and denote its arc flows by $x_{ij}^{(1)}$. Let another flow pattern with v_2 be given and denote its arc flow by $x_{ij}^{(2)}$. If we superpose the two flow patterns, then we get a flow pattern with value $v_1 + v_2$. Let $x_{ij}^{(3)}$ be the arc flow of this new flow pattern. Then

$$x_{ij}^{(3)} = x_{ij}^{(1)} + x_{ij}^{(2)} \ ,$$

if $x_{ij}^{(1)}$ and $x_{ij}^{(2)}$ are of the same directions, and

$$x_{ij}^{(3)} = \left|x_{ij}^{(1)}\right| - \left|x_{ij}^{(2)}\right| ,$$

if $x_{ij}^{(1)}$ and $x_{ij}^{(2)}$ are of the opposite directions and $x_{ij}^{(1)}$ is of the greater magnitude. We say that two flow patterns are conformal if and only if

$$x_{ij}^{(3)} = x_{ij}^{(1)} + x_{ij}^{(2)}$$

for all arcs.

A particular flow pattern, called a "flow path," is a flow pattern with $x_{s1} = x_{12} = \cdots = x_{nt} = 1$. If the cost of a flow pattern with value v is known, and we superpose a flow path on this given flow pattern, the resulting pattern has value v + 1. The total cost of the flow pattern with value v + 1 is the sum of the cost of the flow pattern with value v, plus the sum of u_{ij} and d_{ij} used in the flow path. The sum of u_{ij} and d_{ij} used in the flow path is called the incremental cost of the flow path.

2. ALGORITHM

The algorithm for solving the minimum convex-cost flow problem can be simply described as follows.

Starting with all $x_{ij} \equiv 0$, send one unit of flow from N_s to N_t along the path whose incremental cost relative to the existing flow pattern is minimum. (This can be done by many of the existing shortest-path methods with u_{ij} and d_{ji} as the lengths; see, for example, [6], [10]). Redefine

the u_{ij} and d_{ji} based on the new flow pattern obtained, and send one additional unit of flow along the path with minimum incremental cost. The process of using the minimum incremental cost path is repeated until the total outflow of N_s is v (or the total inflow of N_t is v).

Many proofs are known for the case where the objective function (1) is a linear function. In Beale [1], an algorithm is given for a bipartite network with convex cost, and it starts with a feasible solution. It is easy to convert the existing proofs and ideas into the case of an arbitrary network and convex-cost functions, and to show that at every successive stage of the algorithm, the flow pattern is optimum for the corresponding parameter v. This will be discussed later.

Let us give one example to illustrate the algorithm. Consider Fig. 1. The cost function of each arc is $c_{ij}\, x_{ij}^2$, with c_{ij} written beside the arc and $c_{ij} = c_{ji}$.

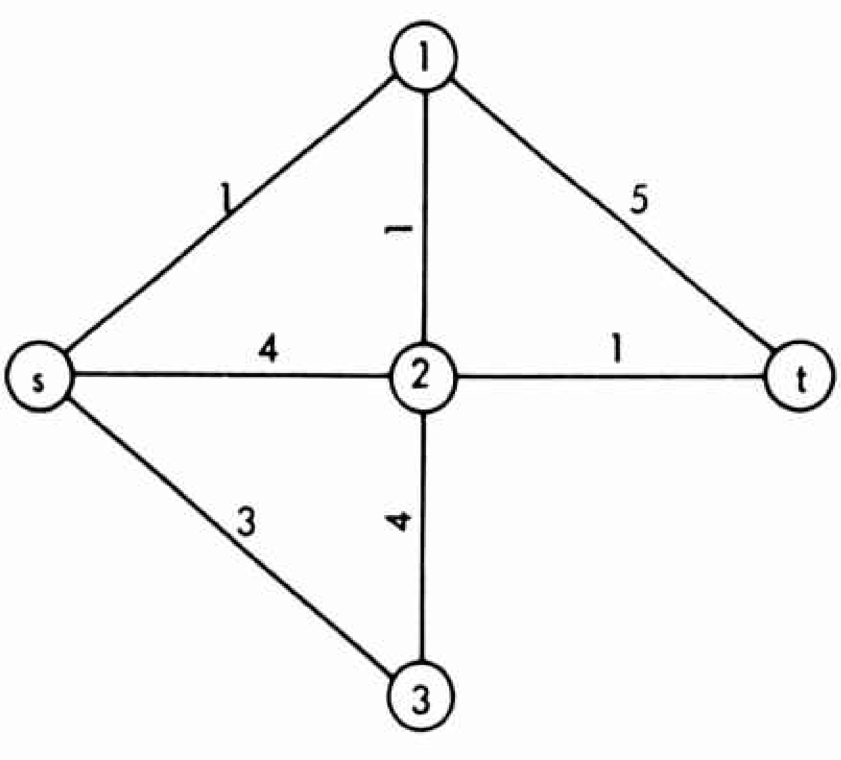

Fig. 1

Assume a given flow pattern as shown in Fig. 2.

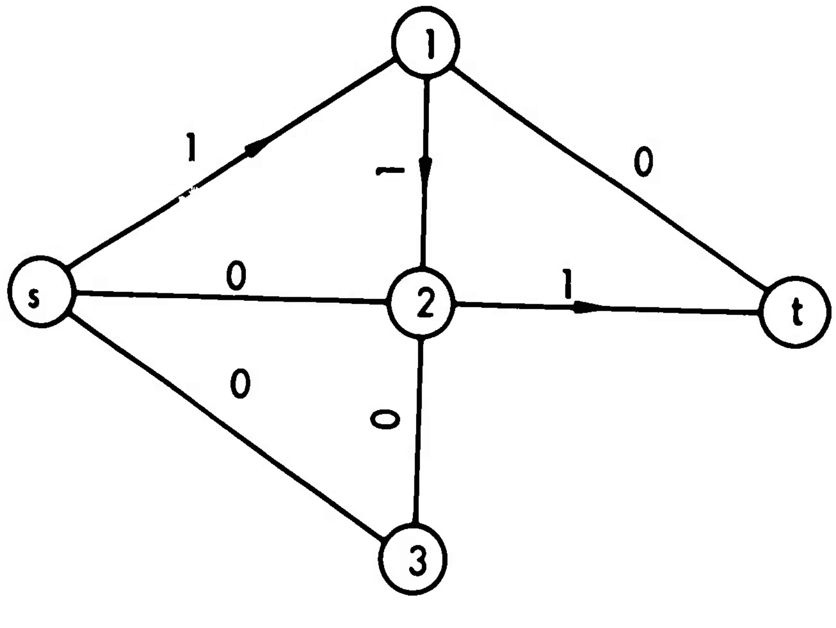

Fig. 2

Then the up-costs and down-costs of every arc are calculated from (3) and (4) with the result shown in Fig. 3, where the first number beside an arc is the up-cost, the second number is the down-cost, and the directions of up-costs are the same as in the flow pattern of Fig. 2.

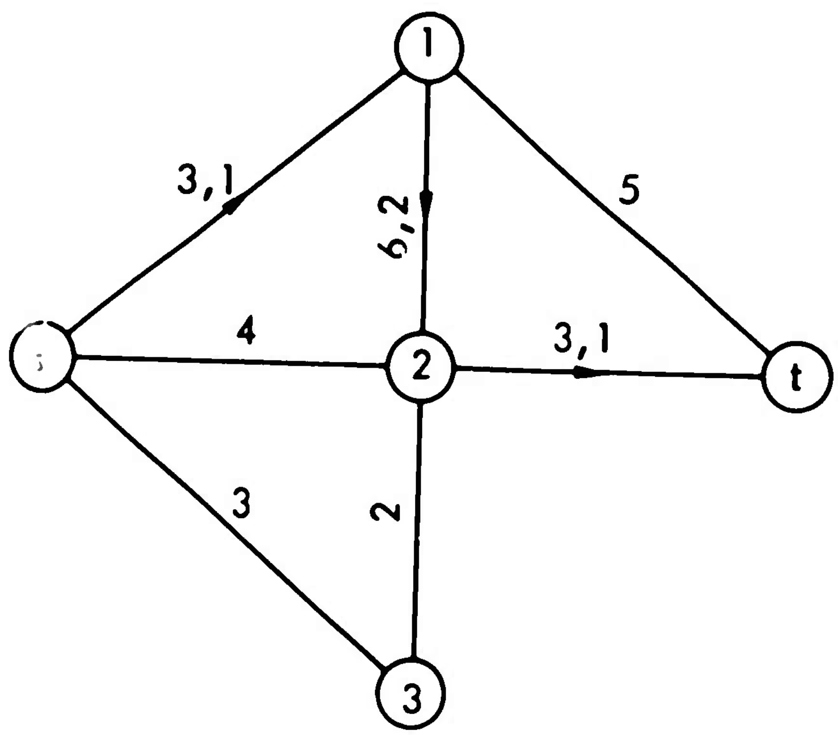

Fig. 3

For example, the up-cost of A_{12} is $2 \cdot 2^2 - 2 \cdot 1^2 = 6$. If an arc has no flow, like A_{s2}, the cost of flow from both directions will be $4 \cdot 1^2 - 4 \cdot 0^2 = 4$. If we want tc send one additional unit flow with minimum incremental cost, we should use the arcs A_{s2}, A_{21}, and A_{1t} with total incremental cost $4 + (-2) + 5 = 7$, with the resulting flow pattern as shown in Fig. 4.

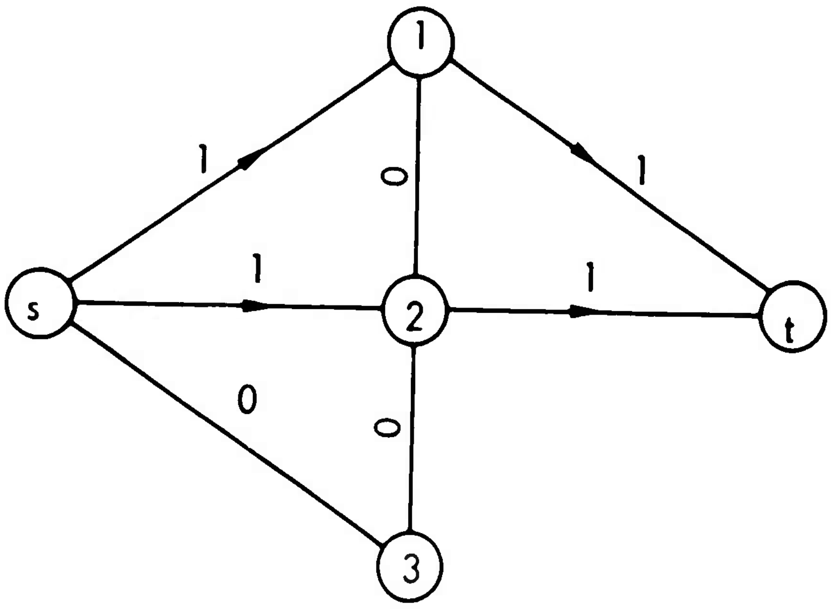

Fig. 4

3. APPLIC...IONS

3.1. Maximum-Flcw Min-Cut Theorem

The problem of finding maximum flow through a network with b_{ij} as the branch capacities of arcs can be formulated as follows:

$$\text{Max } v$$

subject to

(7) $$\sum_i x_{ij} - \sum_k x_{jk} = \begin{cases} -v & \text{for } j = s , \\ 0 & \text{for } j \neq s,t , \\ v & \text{for } j = t, \end{cases}$$

and

(8) $$0 \leq x_{ij} \leq b_{ij} \quad \text{for all } i,\ j.$$

This problem can be formulated as a minimum convex-cost flow problem as in (1) and (2) by defining

$$u_{ij}(x_{ij}) = 0 \qquad \text{if } x_{ij} < b_{ij} ,$$

$$d_{ji}(x_{ij}) = 0 \qquad \text{if } x_{ij} \geq 0 ,$$

$$u_{ij}(x_{ij}) = \infty \qquad \text{if } x_{ij} = b_{ij} ,$$

and min $z = \Sigma\, c_{ij}(x_{ij})$ subject to equation (2). The cost function of an arc is shown in Fig. 5.

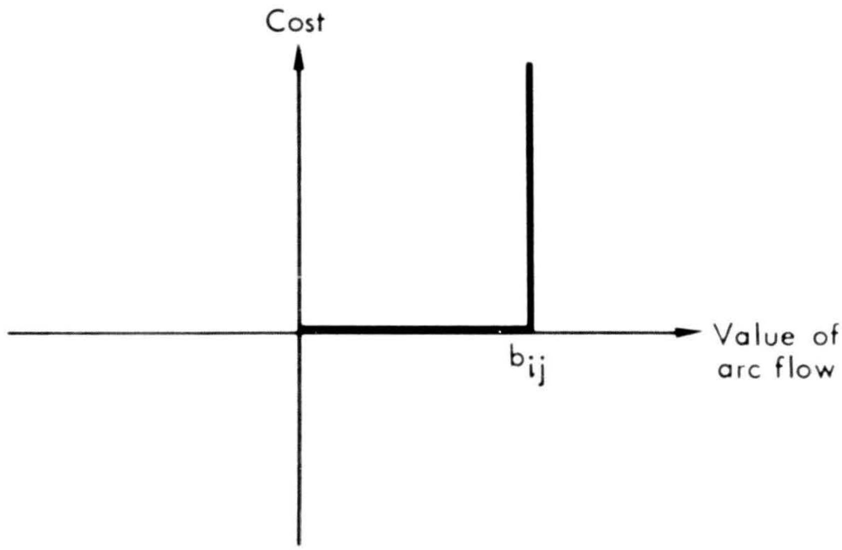

Fig. 5

The v in (2) is taken as a parameter. The maximum flow v is obtained when the value of the objective function z becomes infinity for v + 1. This means when the value in (2) is v, there is no arc in the network with infinity cost, and when the value is v + 1, at least one arc is with infinity cost. Since we are minimizing z, there must be a set of arcs which form a cut in the flow pattern with value v in which all $x_{ij} = b_{ij}$. Hence, the value v is the maximum-flow value. A proof as well as an algorithm is shown in Busacker and Gower [3]. We can generalize the approach used in [3] to solve the following case:

$$\text{(9)} \qquad \min z = \Sigma\, c_{ij}(x_{ij})$$

subject to

$$\text{(10)} \qquad \sum_i x_{ij} - \sum_k x_{jk} = \begin{cases} -v & j = s\ , \\ 0 & j \neq j,t\ , \\ v & j = t\ , \end{cases}$$

$$\text{(11)} \qquad 0 \leq x_{ij} \leq b_{ij}\ ,$$

where in (9) z is a sum of any convex-cost functions. The algorithm of always choosing the flow path with minimum incremental cost works for the case when the objective function is a linear function (see, for example, [3]), and its validity does not depend on how many arcs connect the two nodes. We shall transform equations (9), (10), and (11) into a linear case as follows.

Consider an arc b_{ij} as a set of arcs, each with unit branch capacity. Index those arcs with positive integers

$1,2,\ldots,p$. The cost of the k-th arc from N_i to N_j is

$$c_{ij}(k) - c_{ij}(k-1) \quad \text{if } x_{ij} = 0 ,$$

and the cost is ∞ if $x_{ij} = 1$. It follows from the convexity of $c_{ij}(x_{ij})$ that the up-costs of arcs from N_i to N_j are always monotonically increasing with the index of the arcs, while the down-cost of the k-th arc is negative if it has flow. Assume that $x_{ij} > 0$; then if we want to send additional flow from N_i to N_j, we always use the arc with smallest index if that arc is not saturated; if we want to send flow from N_j to N_i, we always use the saturated arc with largest index. Then (9) becomes a set of linear cost functions of those unit-capacity arcs, since in no case would we use an arc with infinity cost.

3.2. Increasing the Capacity of a Network

This problem solved in [8] can be stated as follows. A network with branch capacity b_{ij} is given. Now, we want to increase the branch capacity or construct new arcs such that the maximum flow from N_s to N_t is increased. The cost of increasing or constructing a unit branch capacity from N_i to N_j is c_{ij}. The problem is to find

$$\text{Max } z = V$$

subject to $c = c_{ij}\, y_{ij}$

$$\sum_i x_{ij} - \sum_k x_{jk} = \begin{cases} -v \text{ for } j = s , \\ 0 \text{ for } j \neq s,t , \\ v \text{ for } j = t , \end{cases}$$

$$0 \leq x_{ij} < b_{ij} + y_{ij} \; ;$$

i.e., with a given budget, we want to maximize the flow from N_s to N_t by allocating y_{ij} appropriately.

This problem also can be solved by the algorithm of minimal incremental cost path. Let us define the up-cost and down-cost of arc flow as follows:

$$u_{ij}(x_{ij}) = 0 \qquad \text{if } x_{ij} < b_{ij} ,$$

$$u_{ij}(x_{ij}) = c_{ij} \qquad \text{if } x_{ij} \geq b_{ij} ,$$

$$d_{ji}(x_{ij}) = 0 \qquad \text{if } 0 \leq x_{ij} \leq b_{ij} ,$$

$$d_{ji}(x_{ij}) = - c_{ij} \text{ if } x_{ij} > b_{ij} .$$

Then the solution is to always send from N_s to N_t one unit of flow along the minimal incremental cost path (since we can consider the problem as min $\Sigma\, c_{ij}, y_{ij}$ and treat v as a parameter), until the total amount of money used up is c. Then

$$y_{ij} = x_{ij} - b_{ij} \quad \text{if } x_{ij} > b_{ij} ,$$

$$y_{ij} = 0 \qquad \text{if } x_{ij} \leq b_{ij} .$$

<u>3.3. Electrical Network</u>

Consider a passive electrical network with one current-input source at N_s and one current-output source at N_t. From Ohm's law, the electrical current x_{ij} from N_i to N_j is proportional to the potential difference ϕ_{ij} and inversely proportional to the resistance r_{ij} of that arc, i.e.,

$$x_{ij} = \frac{\phi_{ij}}{r_{ij}} .$$

The work done by that arc is $x_{ij}\, \phi_{ij} = r_{ij}\, x_{ij}^2$. To solve an electrical network of the above type, we can solve the simultaneous equations given by Kirchoff's node law and Kirchoff's loop law. Alternatively, we can regard Kirchoff's node law as linear constraints of the currents x_{ij}, and minimize the total work done. This then becomes the following quadratic programming problem (see [5]):

$$\text{Min } z = r_{ij}\, x_{ij}^2$$

subject to

$$\sum_i x_{ij} - \sum_j x_{jk} = \begin{cases} -v & \text{for } j = s\ , \\ 0 & \text{for } j \neq s,t\ , \\ v & \text{for } j = t\ . \end{cases}$$

This problem can again be handled by the minimum incremental cost-path algorithm by defining costs of arcs as follows: Let

$$z = r_{ij} x_{ij}^2\ ,$$

where

$$u_{ij}(x_{ij}) = r_{ij}(2x_{ij} + 1) \qquad \text{if } x_{ij} \geq 0\ ,$$

$$d_{ji}(x_{ij}) = -r_{ij}(2x_{ij} - 1) \qquad \text{if } x_{ij} \geq 1\ .$$

For given strengths of inflow current source and outflow current sinks, we send the current along the minimum incremental cost path for source to the sink until the inflow rate is v.

3.4. Laplace Equations

The consideration of a network problem as a boundary-value problem was done in [2]. Let us consider the Laplace equation in a region G:

(12) $\nabla^2 \phi = 0$

with $\frac{\partial\phi}{\partial n}$ prescribed on the boundary of G. If we use difference equations to replace (12) and use a uniform grid, then the value of ϕ_0 at a point is the average value of its four neighbors (see Figs. 6a and 6b), i.e.,

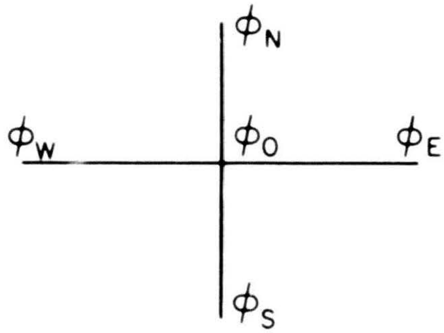

Fig. 6a

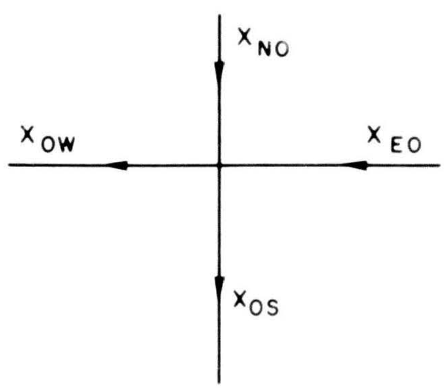

Fig. 6b

(13) $\qquad 4\phi_0 - \phi_N - \phi_S - \phi_E - \phi_W = 0.$

Rewriting (13) and letting $x_{E0} = \phi_E - \phi_0$, etc., we have

(14) $\qquad x_{E0} - x_{0W} + x_{N0} - x_{0S} = 0.$

Equation (14) then can be considered as the conservation-of-flow equation, with x_{E0} the arc flow from node E to node 0. The boundary condition of prescribing $\frac{\partial \phi}{\partial n}$ is then interpreted as the condition of inflow and outflow at sources and sinks in a network. The Dirichlet principle (see for example [9]) for solving a Laplace equation can then be regarded as that of minimizing a quadratic objective function,

(15) $\qquad \min z = \Sigma\, x_{ij}^2 ,$

subject to equation (14) at interior points of region G and satisfying the boundary condition $x_{ij} = \frac{\partial \phi}{\partial n}$ at the boundary of G.

We have the objective function

$$z = \Sigma\, x_{ij}^2 ,$$

where

$$u_{ij}(x_{ij}) = 2x_{ij} + 1 \qquad \text{if } x_{ij} \geq 0,$$

$$d_{ji}(x_{ij}) = -\,2x_{ij} + 1 \qquad \text{if } x_{ij} \geq 1.$$

Then the Laplace equation can be solved by min incremental cost path from sources to sinks, as done previously. Special examples can be given to show that this approach is better.

4. DISCUSSION

We have assembled several problems in different areas and shown that all of them can be solved by algorithms of shortest-path type. The distance functions (or cost functions of arcs), however, are not fixed but depend on the current existing flows in the network, and can be negative. Hence, we must use iterative-type, shortest-path algorithms such as in [6], [10]. As the flow patterns always maintain their optimality, there will never be any cycle of arcs in the network for which the total cost of going around the cycle is negative.

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